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Asymptotics in nonlinear evolution system with dissipation and ellipticity on quadrant ☆

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Abstract

In this paper, we consider an initial boundary value problem for some nonlinear evolution system with dissipation and ellipticity. We establish the global existence and furthermore obtain the L^p $(p \geqslant 2)$ decay rates of solutions corresponding to diffusion waves. The analysis is based on the energy method and pointwise estimates.

Keywords: Nonlinear evolution system; Decay rate; Energy method; Pointwise estimates; A priori estimates

1. Introduction

In this paper, we consider the following nonlinear evolution system with dissipation and ellipticity on the quadrant $\mathbb{R}_+ \times \mathbb{R}_+$:

$$\begin{cases} \psi_t = -(1-\alpha)\psi - \theta_x + \alpha\psi_{xx}, \\ \theta_t = -(1-\alpha)\theta + \nu\psi_x + 2\psi\theta_x + \alpha\theta_{xx}, \end{cases} \tag{1.1}$$

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with initial data

$$(\psi,\theta)(x,0)=(\psi_0,\theta_0)(x),\quad x\geqslant 0, \tag{1.2}$$

and the zero Dirichlet boundary condition

$$(\psi,\theta)(0,t)=(0,0),\quad t\geqslant 0. \tag{1.3}$$

Here it is assumed that $(\psi_0,\theta_0)(0)=(0,0)$ and $\lim_{x\to\infty}(\psi_0,\theta_0)(x)=(\psi_+,\theta_+)$.

The nonlinear interaction between dissipation and ellipticity plays a vital role in a number of physical systems, such as dynamic phase transitions, superposed fluids, Rayleigh–Benard problem, Taylor–Couette instability and fluid flow down an inclined plane, etc. [5,8,9,14,17]. These systems are far from well understood. A simplified system was thus proposed by Hsieh et al. [7]:

$$\begin{cases} \psi_t=-(\sigma-\alpha)\psi-\sigma\theta_x+\alpha\psi_{xx},\\ \theta_t=-(1-\beta)\theta+\nu\psi_x+2\psi\theta_x+\beta\theta_{xx}, \end{cases} \tag{1.4}$$

where α, β, σ and ν are positive constants such that $\alpha<\sigma$ and $\beta<1$. System (1.4) is reduced to (1.1) when $\sigma=1,\alpha=\beta$. Studies on this system are expected to yield insight into aforementioned physical systems with the similar mechanism.

To explain the complexity in system (1.4) by a rough argument, we ignore the damping and diffusion terms temporarily to obtain the linear system

$$\begin{pmatrix}\psi_t\\ \theta_t\end{pmatrix}=\begin{pmatrix}0 & -\sigma\\ \nu & 2\psi\end{pmatrix}\begin{pmatrix}\psi_x\\ \theta_x\end{pmatrix}. \tag{1.5}$$

It is elliptic for $|\psi|<\sqrt{\sigma\nu}$ and hyperbolic for $|\psi|>\sqrt{\sigma\nu}$. Around the zero equilibrium, the small disturbance trigers the growth in $|\psi|$ because of the inherent instability of ellipticity. However, at a position where $|\psi|>\sqrt{\sigma\nu}$, system (1.5) becomes hyperbolic. Correspondingly in (1.4), dissipation terms tend to draw the system back to the equilibrium. For suitable coefficients, a “switching back and forth” phenomenon is expected due to the interplay among ellipticity, hyperbolicity and dissipation. But this phenomenon will not occur in the analysis of this paper since we will only consider small solutions so that the linear system (1.5) is always elliptic. Numerical investigations have also revealed exciting behaviors of the system (1.4) [6,7]. Depending on the parameters, the system may admit spikes, periodic and quasiperiodic solutions, even chaos.

Next let us recall some theoretical results on system (1.4). Based on the Fourier analysis and the energy method, the Cauchy problem of (1.4) with $\sigma=1$ and $\alpha=\beta$ was studied in [16], where the global existence, nonlinear stability and optimal decay rates of solutions were obtained for coefficients with $\nu<4\alpha(1-\alpha)$. Boundedness of initial data is assumed in the sense of

$$(\psi_0,\theta_0)(x)\in L^2\cap W^{1,\infty}\big(\mathbb{R},\mathbb{R}^2\big), \tag{1.6}$$

which implies that ψ_0 and θ_0 vanish at infinity. For initial data with different end states:

$$\lim_{x\to\pm\infty}(\psi_0,\theta_0)(x)=(\psi_\pm,\theta_\pm),\qquad (\psi_+,\theta_+)\neq(\psi_-,\theta_-),$$

the global existence and asymptotics to the Cauchy problem were obtained by constructing a correct function and using the energy method [21]. More precisely, it was proved that the small perturbation make solutions decay exponentially around diffusion waves in the following system obtained by the Darcy’s law [3,13]:

$$\begin{cases} \bar\psi_t=-(1-\alpha)\bar\psi-\bar\theta_x+\alpha\bar\psi_{xx},\\ -(1-\alpha)\bar\theta+\nu\bar\psi_x=0. \end{cases} \tag{1.7}$$

Similar results were obtained in [2] for the convergence toward another kind of diffusion waves defined by

$$\begin{cases} \bar{\psi}_t = -(1-\alpha)\bar{\psi} + \alpha\bar{\psi}_{xx}, \\ \bar{\theta}_t = -(1-\alpha)\bar{\theta} + \alpha\bar{\theta}_{xx}. \end{cases} \tag{1.8}$$

In this paper, we shall investigate the global existence and asymptotics of the initial boundary value problem (1.1)–(1.3). Generally speaking, the presence of the boundary layer requires some special treatment in making estimates [4,10–12,18,19]. Here we will consider the zero Dirichlet boundary condition instead of the general Dirichlet boundary condition to avoid the difficulties that arise due to a boundary layer. Precisely, for the zero Dirichlet boundary condition, we use the Green function on the half-space to obtain the representation of solutions to the derivation from diffusion waves (1.8) and furthermore apply the energy method to get some a priori estimates. The global existence of solutions follows from the standard argument that the local existence and uniform a priori estimates yield the global existence. This paper is the first step to deal with the general Dirichlet boundary condition.

The rest of this paper is arranged as follows. In Section 2, we study linear diffusion waves and make decay estimates of solutions and their derivatives. In Section 3, the global existence is established through the local existence and a priori estimates. In Section 4, we get the L^2 decay rate of the deviation from the diffusion wave through another a priori estimate. Furthermore, we obtain L^p $(p \geqslant 2)$ decay rates by pointwise estimates. Some concluding remarks are made in Section 5.

Notations. Throughout this paper, we denote any positive constants by C, without making ambiguity. The Lebesgue space on $\mathbb{R}_+$, $L^p = L^p(\mathbb{R}_+)$ $(1 \leqslant p \leqslant \infty)$ is endowed with the norm $\|f\|_{L^p} = (\int_{\mathbb{R}_+} |f(x)|^p\,dx)^{1/p}$, $1 \leqslant p < \infty$, or $\|f\|_{L^\infty} = \sup_{\mathbb{R}_+} |f(x)|$. When $p = 2$, we write $\|\cdot\|_{L^2(\mathbb{R}_+)} = \|\cdot\|$. The lth order Hilbert space $H^l(\mathbb{R}_+)$ is endowed with the norm $\|f\|_{H^l(\mathbb{R}_+)} = \|f\|_l = (\sum_{i=0}^{l} \|\partial_x^i f\|^2)^{1/2}$. For simplicity, $\|f(\cdot,t)\|_{L^p}$ and $\|f(\cdot,t)\|_l$ are denoted by $\|f(t)\|_{L^p}$ and $\|f(t)\|_l$, respectively.

2. Diffusion waves

In this section, we construct diffusion waves corresponding to the nonlinear system (1.1). What is more, the decay rates of diffusion waves are obtained with the help of their explicit representations.

As in [3,13], we expect that solutions of (1.1) converge toward those of the following linear system:

$$\begin{cases} \bar{\psi}_t = -(1-\alpha)\bar{\psi} + \alpha\bar{\psi}_{xx}, \\ \bar{\theta}_t = -(1-\alpha)\bar{\theta} + \alpha\bar{\theta}_{xx}. \end{cases} \tag{2.1}$$

By setting $\bar{\psi}(x,t) = \bar{\phi}(x,t)e^{-(1-\alpha)t}$, we can derive a heat equation from $(2.1)_1$

$$\bar{\phi}_t = \alpha\bar{\phi}_{xx}. \tag{2.2}$$

A self-similarity solution $\bar{\phi}(x,t)$ takes the following form:

$$\bar{\phi}(x,t) = p(\xi) = p\left(\frac{x}{\sqrt{1+t}}\right), \quad 0 < \xi < \infty, \tag{2.3}$$

where $\xi = \frac{x}{\sqrt{1+t}}$. From (2.2) and (1.3), we have

$$\begin{cases} -\frac{1}{2}\xi p'(\xi) = \alpha p''(\xi), \\ p(0) = 0, \quad p(\infty) = \psi_+. \end{cases} \tag{2.4}$$

We get by the direct calculation that

$$\bar{\phi}(x,t) = p(\xi) = \psi_+ - \frac{2\psi_+}{\sqrt{4\pi\alpha(1+t)}} \int_x^\infty \exp\left(-\frac{y^2}{4\alpha(1+t)}\right) dy.$$

Diffusion waves defined by (2.1) then read

$$\begin{cases} \bar{\psi}(x,t) = \psi_+ e^{-(1-\alpha)t}\bigl(1 - 2\int_x^\infty G(y,t+1)\,dy\bigr), \\ \bar{\theta}(x,t) = \theta_+ e^{-(1-\alpha)t}\bigl(1 - 2\int_x^\infty G(y,t+1)\,dy\bigr), \end{cases} \tag{2.5}$$

where $G(x,t) = \frac{1}{\sqrt{4\pi\alpha t}}\exp\{-\frac{x^2}{4\alpha t}\}$ is the heat kernel function.

It is easy to show that

$$\begin{cases} (\bar{\psi},\bar{\theta})(x,t) \to (\psi_+ e^{-(1-\alpha)t}, \theta_+ e^{-(1-\alpha)t}), & x \to \infty, \\ (\bar{\psi},\bar{\theta})(0,t) = (0,0), & t \geqslant 0. \end{cases} \tag{2.6}$$

On the other hand, as in [3,13,20], system (1.1) implies that

$$\psi(x,t) \to \psi_+ e^{-(1-\alpha)t}, \quad x \to \infty, \tag{2.7}$$

and

$$\theta(x,t) \to \theta_+ e^{-(1-\alpha)t}, \quad x \to \infty. \tag{2.8}$$

Therefore we have

$$\begin{cases} (\psi - \bar{\psi}, \theta - \bar{\theta})(x,t) \to (0,0), & x \to \infty, \\ (\psi - \bar{\psi}, \theta - \bar{\theta})(0,t) = (0,0), & t \geqslant 0. \end{cases} \tag{2.9}$$

Now we consider the asymptotics of $\bar{\psi}(x,t)$, $\bar{\theta}(x,t)$ and their derivatives in $L^p(\mathbb{R}_+)$. First, the heat kernel function has the following property.

Lemma 2.1. *When $1 \leqslant p \leqslant +\infty$, $0 \leqslant l, k < +\infty$, it holds that*

$$\bigl\|\partial_t^l \partial_x^k G(t)\bigr\|_{L^p} \leqslant C t^{-\frac{1}{2}(1-\frac{1}{p})-l-\frac{k}{2}}.$$

By standard calculations, it is readily shown that diffusion waves $(\bar{\psi},\bar{\theta})$ have the following properties.

Lemma 2.2. *For functions $(\bar{\psi},\bar{\theta})(x,t)$ defined by* (2.5)*, it holds that*

(i) $\|\partial_t^l \bar{\psi}(t)\|_{L^\infty} \leqslant Ce^{-(1-\alpha)t}$, $\|\partial_t^l \bar{\theta}(t)\|_{L^\infty} \leqslant Ce^{-(1-\alpha)t}$, $l = 0,1,2,\ldots;$
(ii) *for any p with $1 \leqslant p \leqslant +\infty$,*

$$\bigl\|\partial_t^l \partial_x^k \bar{\psi}(t)\bigr\|_{L^p} \leqslant C|\psi_+|e^{-(1-\alpha)t}(1+t)^{\frac{1}{2p}-\frac{k}{2}}, \quad k = 1,2,\ldots,\ l = 0,1,2,\ldots,$$

$$\bigl\|\partial_t^l \partial_x^k \bar{\theta}(t)\bigr\|_{L^p} \leqslant C|\theta_+|e^{-(1-\alpha)t}(1+t)^{\frac{1}{2p}-\frac{k}{2}}, \quad k = 1,2,\ldots,\ l = 0,1,2,\ldots.$$

3. Global existence

3.1. Reformulation and main results

Deviation from diffusion waves

$$\begin{cases} u(x,t) = \psi(x,t) - \bar{\psi}(x,t), \\ v(x,t) = \theta(x,t) - \bar{\theta}(x,t) \end{cases} \tag{3.1}$$

is governed by

$$\begin{cases} u_t = -(1-\alpha)u - v_x + \alpha u_{xx} - \bar{\theta}_x, \\ v_t = -(1-\alpha)v + \nu u_x + 2uv_x + \alpha v_{xx} + 2\bar{\psi} v_x + 2\bar{\theta}_x u + F(x,t), \\ \quad x > 0,\ t > 0, \end{cases} \tag{3.2}$$

with initial data

$$(u, v)(x, 0) = (u_0, v_0)(x), \quad x \geqslant 0, \tag{3.3}$$

and the boundary condition

$$(u, v)(0, t) = (0, 0), \quad t \geqslant 0. \tag{3.4}$$

Here we have used the following notations:

$$\begin{cases} u_0(x) = \psi_0(x) - \bar{\psi}(x, 0) \to 0, \quad \text{as } x \to \infty, \\ v_0(x) = \theta_0(x) - \bar{\theta}(x, 0) \to 0, \quad \text{as } x \to \infty, \\ F(x,t) = \nu \bar{\psi}_x + 2\bar{\psi}\bar{\theta}_x. \end{cases} \tag{3.5}$$

We seek solutions to the problem (3.2)–(3.4) in the set of functions

$$X(0,T) = \big\{(u,v) \,\big|\, u, v \in L^\infty\big(0,T; H^1(\mathbb{R}_+)\big) \cap L^2\big(0,T; H^2(\mathbb{R}_+)\big)\big\}, \tag{3.6}$$

where $0 < T \leqslant \infty$. Now we state our main result as follows.

Theorem 3.1. *Let* $(u_0, v_0)(x) \in H^1(\mathbb{R}_+, \mathbb{R}^2)$, $0 < \alpha < 1$ *and* $0 < \nu < 4\alpha(1-\alpha)$. *Assume that both* $\delta = |\psi_+| + |\theta_+|$ *and* $\delta_0 = \|u_0\|_1^2 + \|v_0\|_1^2$ *are sufficiently small. Then there exist unique global solutions* $(u, v)(x,t) \in X(0,\infty)$ *to the initial boundary value problem* (3.2)–(3.4)*, which satisfy*

$$\big\|u(t)\big\|_1^2 + \big\|v(t)\big\|_1^2 + \int_0^t \big(\big\|u(\tau)\big\|_2^2 + \big\|v(\tau)\big\|_2^2\big)\,d\tau \leqslant C(\delta + \delta_0)^{\frac{1}{2}}, \quad \text{for any } t \geqslant 0, \tag{3.7}$$

and

$$\sup_{x \in \mathbb{R}_+} \big|(u,v)(x,t)\big| \to 0 \quad \text{as } t \to \infty. \tag{3.8}$$

Furthermore, for any $2 \leqslant p \leqslant \infty$ *and* $t \geqslant \tau > 0$, *it holds that*

$$\sum_{k=0}^{1} \big(\big\|\partial_x^k u(t)\big\|_{L^p} + \big\|\partial_x^k v(t)\big\|_{L^p}\big) \leqslant C(\tau) e^{-\frac{l}{2}t}, \tag{3.9}$$

where the positive constant l *is defined by* (4.2) *in Section* 4 *and* $C(\tau)$ *is a positive constant depending only on* τ.

3.2. Local existence

In order to get the explicit representation of solutions to system (3.2), let us first consider the initial boundary value problem

$$\begin{cases} \bar{u}_t = -(1-\alpha)\bar{u} + \alpha\bar{u}_{xx}, \\ \bar{v}_t = -(1-\alpha)\bar{v} + \alpha\bar{v}_{xx}, \quad x > 0,\ t > 0, \\ (\bar{u},\bar{v})(0,t) = (0,0), \quad t \geqslant 0, \\ (\bar{u},\bar{v})(x,0) = (\bar{u}_0,\bar{v}_0)(x), \quad x \geqslant 0. \end{cases} \tag{3.10}$$

By introducing the following transformation:

$$\bar{u}(x,t) = w(x,t)e^{-(1-\alpha)t}, \qquad \bar{v}(x,t) = z(x,t)e^{-(1-\alpha)t},$$

it is easy to see that system (3.10) has solutions

$$\begin{cases} \bar{u}(x,t) = e^{-(1-\alpha)t}\int_0^\infty (G(x-y,t) - G(x+y,t))\bar{u}_0(y)\,dy \\ \qquad = \frac{e^{-(1-\alpha)t}}{\sqrt{4\pi\alpha t}}\int_0^\infty \big(e^{-\frac{(x-y)^2}{4\alpha t}} - e^{-\frac{(x+y)^2}{4\alpha t}}\big)\bar{u}_0(y)\,dy, \\ \bar{v}(x,t) = e^{-(1-\alpha)t}\int_0^\infty (G(x-y,t) - G(x+y,t))\bar{v}_0(y)\,dy \\ \qquad = \frac{e^{-(1-\alpha)t}}{\sqrt{4\pi\alpha t}}\int_0^\infty \big(e^{-\frac{(x-y)^2}{4\alpha t}} - e^{-\frac{(x+y)^2}{4\alpha t}}\big)\bar{v}_0(y)\,dy. \end{cases} \tag{3.11}$$

Thus we define an integral operator $K(t)*$ by

$$\begin{aligned} \big(K(t)*f\big)(x) &= e^{-(1-\alpha)t}\int_0^{+\infty} \big(G(x-y,t) - G(-x-y,t)\big)f(y)\,dy \\ &= e^{-(1-\alpha)t}\big(G(\cdot,t)*f\big)(x) - e^{-(1-\alpha)t}\big(G(\cdot,t)*f\big)(-x), \end{aligned} \tag{3.12}$$

for any $x \geqslant 0$ and any function $f(x)$. Hence $K(t)*f$ is the difference between two convolutions. From Lemma 2.1, we have the following estimates about the integral operator.

Lemma 3.2. *When $1 \leqslant p \leqslant +\infty$, $0 \leqslant l, k < +\infty$, we have*

$$\big\|\partial_t^l\partial_x^k K(t)\big\|_{L^p} \leqslant Ce^{-(1-\alpha)t}t^{-\frac{1}{2}(1-\frac{1}{p})-l-\frac{k}{2}}. \tag{3.13}$$

In view of the Duhamel's principle, we can rewrite (3.2)–(3.4) in the following integral form:

$$\begin{cases} u(x,t) = \big(K(t)*u_0\big)(x) + \int_0^t K(t-s)*(-v_x - \bar{\theta}_x)(\cdot,s)\,ds, \\ v(x,t) = \big(K(t)*v_0\big)(x) + \int_0^t K(t-s)*(vu_x + 2uv_x + 2\bar{\psi}v_x + 2\bar{\theta}_x u + F)(\cdot,s)\,ds. \end{cases} \tag{3.14}$$

Hence we can construct the approximation sequences of solutions and obtain the local existence by implementing the standard arguments with the Brower fixed point principle, cf. [1].

Lemma 3.3 *(Local existence). If $(u_0,v_0)(x) \in H^1(\mathbb{R}_+,\mathbb{R}^2)$, then there exists $t_0 > 0$ depending only on $\|(u_0,v_0)(x)\|_{H^1(\mathbb{R}_+,\mathbb{R}^2)}$, such that the initial boundary value problem* (3.2)–(3.4) *admits unique smooth solutions $(u,v)(x,t) \in X(0,t_0)$ satisfying*

$$\big\|(u,v)(\cdot,t)\big\|_{H^1(\mathbb{R}_+,\mathbb{R}^2)} \leqslant 2\big\|(u_0,v_0)(x)\big\|_{H^1(\mathbb{R}_+,\mathbb{R}^2)}, \quad 0 \leqslant t \leqslant t_0. \tag{3.15}$$

3.3. Global existence

In order to get the global existence, we must show that solutions obtained in Lemma 3.3 are bounded from above by a positive constant that depends only on the initial data.

Throughout this subsection, we suppose that $(u, v)(x, t) \in X(0, T)$ are solutions to the problem (3.2)–(3.4). Next let us devote ourselves to making estimates of (u, v) under the following a priori assumption:

$$N(T) = \sup_{0 \leqslant t \leqslant T} \left\{ \sum_{k=0}^{1} \left(\left\| \partial_x^k u(t) \right\|^2 + \left\| \partial_x^k v(t) \right\|^2 \right) \right\} \leqslant \delta_1^2, \tag{3.16}$$

where $0 < \delta_1 \ll 1$.

From the Sobolev inequality $\|f\|_{L^\infty} \leqslant \|f\|^{\frac{1}{2}} \|f_x\|^{\frac{1}{2}}$, we have

$$\left\| (u, v)(x, t) \right\|_{L^\infty([0,T] \times \mathbb{R}_+, \mathbb{R}^2)} \leqslant \delta_1. \tag{3.17}$$

Moreover, if $\nu < 4\alpha(1 - \alpha)$ as in [16], we can find $\varepsilon \in (0, 2)$, $c_0 > 0$ such that

$$\begin{cases} 2c_0\alpha - \frac{1}{(1-\alpha)\varepsilon} > 0, \\ 2(1 - \alpha) - \frac{c_0\nu^2}{\alpha\varepsilon} > 0. \end{cases} \tag{3.18}$$

In fact, we set $k = \frac{\nu}{4\alpha(1-\alpha)}$, then $k \in (0, 1)$. Choosing

$$\begin{cases} \varepsilon = k + 1, \\ c_0 = \frac{1}{2\alpha(1-\alpha)} \left(\frac{1}{2(k+1)} + \frac{k+1}{8k^2} \right), \end{cases} \tag{3.19}$$

one can easily check that ε and c_0 satisfy (3.18).

Lemma 3.4. *Let the assumptions in Theorem* 3.1 *hold. If δ and δ_1 are sufficiently small, then for any* $0 \leqslant t \leqslant T$, *it holds that*

$$\int_0^\infty \left(u^2 + v^2\right) dx + \int_0^t \int_0^\infty \left(u^2 + v^2\right) dx\, d\tau + \int_0^t \int_0^\infty \left(u_x^2 + v_x^2\right) dx\, d\tau \leqslant C_1(\delta + \delta_0), \tag{3.20}$$

where C_1 is a positive constant depending only on α and ν.

Proof. Multiplying the first equation of (3.2) by $2u$, the second equation by $2c_0 v$, and integrating over $(0, \infty) \times (0, t)$, by the Cauchy–Schwarz inequality we arrive at

$$\begin{aligned} &\int_0^\infty \left(u^2 + c_0 v^2\right) dx + 2(1 - \alpha) \int_0^t \int_0^\infty \left(u^2 + c_0 v^2\right) dx\, d\tau + 2\alpha \int_0^t \int_0^\infty \left(u_x^2 + c_0 v_x^2\right) dx\, d\tau \\ &\quad = \|u_0\|^2 + c_0 \|v_0\|^2 - 2 \int_0^t \int_0^\infty u v_x \, dx\, d\tau + 2\nu c_0 \int_0^t \int_0^\infty v u_x \, dx\, d\tau - 2 \int_0^t \int_0^\infty u \bar{\theta}_x \, dx\, d\tau \\ &\qquad + 4c_0 \int_0^t \int_0^\infty u v v_x \, dx\, d\tau - 2c_0 \int_0^t \int_0^\infty \bar{\psi}_x v^2 \, dx\, d\tau + 4c_0 \int_0^t \int_0^\infty \bar{\theta}_x u v \, dx\, d\tau \end{aligned}$$

$$
\begin{aligned}
&+2c_0\int_0^t\int_0^\infty vF(x,\tau)\,dx\,d\tau\\
&\leqslant (1+c_0)\delta_0+\varepsilon(1-\alpha)\int_0^t\int_0^\infty u^2\,dx\,d\tau+\frac{1}{\varepsilon(1-\alpha)}\int_0^t\int_0^\infty v_x^2\,dx\,d\tau\\
&\quad+\varepsilon\alpha\int_0^t\int_0^\infty u_x^2\,dx\,d\tau+\frac{c_0^2\nu^2}{\varepsilon\alpha}\int_0^t\int_0^\infty v^2\,dx\,d\tau+\delta(1-\alpha)\int_0^t\int_0^\infty u^2\,dx\,d\tau\\
&\quad+\frac{1}{\delta(1-\alpha)}\int_0^t\int_0^\infty \bar\theta_x^2\,dx\,d\tau+2c_0\|u\|_{L^\infty([0,T]\times\mathbb{R}_+)}\int_0^t\int_0^\infty\big(v^2+v_x^2\big)\,dx\,d\tau\\
&\quad+2c_0\|\bar\psi_x\|_{L^\infty([0,T]\times\mathbb{R}_+)}\int_0^t\int_0^\infty v^2\,dx\,d\tau\\
&\quad+2c_0\|\bar\theta_x\|_{L^\infty([0,T]\times\mathbb{R}_+)}\int_0^t\int_0^\infty\big(u^2+v^2\big)\,dx\,d\tau\\
&\quad+c_0\delta\int_0^t\int_0^\infty v^2\,dx\,d\tau+\frac{c_0}{\delta}\int_0^t\int_0^\infty F^2(x,\tau)\,dx\,d\tau.
\end{aligned}\tag{3.21}
$$

Using Lemma 2.2 and (3.17), we derive from the above inequality

$$
\begin{aligned}
&\int_0^\infty\big(u^2+c_0v^2\big)\,dx+\big\{(2-\varepsilon)(1-\alpha)-C\delta\big\}\int_0^t\int_0^\infty u^2\,dx\,d\tau+(2-\varepsilon)\alpha\int_0^t\int_0^\infty u_x^2\,dx\,d\tau\\
&\quad+\left\{2(1-\alpha)-\frac{c_0\nu^2}{\varepsilon\alpha}-C(\delta_1+\delta)\right\}\int_0^t\int_0^\infty c_0v^2\,dx\,d\tau\\
&\quad+\left\{2c_0\alpha-\frac{1}{\varepsilon(1-\alpha)}-C\delta_1\right\}\int_0^t\int_0^\infty v_x^2\,dx\,d\tau\\
&\leqslant C(\delta+\delta_0)+\frac{1}{\delta(1-\alpha)}\int_0^t\int_0^\infty\bar\theta_x^2\,dx\,d\tau+\frac{c_0}{\delta}\int_0^t\int_0^\infty F^2(x,\tau)\,dx\,d\tau.
\end{aligned}\tag{3.22}
$$

For the last two terms on the right-hand side of (3.22), Lemma 2.2 yields

$$
\frac{1}{\delta(1-\alpha)}\int_0^t\int_0^\infty\bar\theta_x^2\,dx\,d\tau\leqslant\frac{1}{\delta(1-\alpha)}C\delta^2\int_0^t e^{-2(1-\alpha)\tau}\,d\tau\leqslant C\delta \tag{3.23}
$$

and

$$\frac{c_0}{\delta}\int_0^t\int_0^\infty F^2(x,\tau)\,dx\,d\tau \leqslant \frac{C}{\delta}\int_0^t\int_0^\infty \big(\bar\psi_x^2+\bar\psi^2\bar\theta_x^2\big)\,dx\,d\tau \leqslant C\delta. \tag{3.24}$$

So we end up with

$$\begin{aligned}
&\int_0^\infty \big(u^2+c_0v^2\big)\,dx + \big\{(2-\varepsilon)(1-\alpha)-C\delta\big\}\int_0^t\int_0^\infty u^2\,dx\,d\tau + (2-\varepsilon)\alpha\int_0^t\int_0^\infty u_x^2\,dx\,d\tau \\
&\quad + \left\{2(1-\alpha)-\frac{c_0\nu^2}{\varepsilon\alpha}-C(\delta+\delta_1)\right\}\int_0^t\int_0^\infty c_0v^2\,dx\,d\tau \\
&\quad + \left\{2c_0\alpha-\frac{1}{\varepsilon(1-\alpha)}-C\delta_1\right\}\int_0^t\int_0^\infty v_x^2\,dx\,d\tau \\
&\leqslant C(\delta+\delta_0).
\end{aligned} \tag{3.25}$$

This ends the proof of Lemma 3.4 with the help of (3.18). □

Lemma 3.5. *Let the assumptions in Theorem* 3.1 *hold. If δ and δ_1 are sufficiently small, then for any $0\leqslant t\leqslant T$, it holds that*

$$\int_0^\infty \big(u_x^2+v_x^2\big)\,dx + \int_0^t\int_0^\infty \big(u_{xx}^2+v_{xx}^2\big)\,dx\,d\tau \leqslant C_2(\delta+\delta_0)^{\frac12}, \tag{3.26}$$

where C_2 is a positive constant depending only on α and ν.

Proof. First, we multiply the first equation of (3.2) by $(-2u_{xx})$, the second equation by $(-2c_0v_{xx})$, respectively, and take integration over $(0,\infty)\times(0,t)$. Then similar to (3.21), we have

$$\begin{aligned}
&\int_0^\infty \big(u_x^2+c_0v_x^2\big)\,dx + 2(1-\alpha)\int_0^t\int_0^\infty \big(u_x^2+c_0v_x^2\big)\,dx\,d\tau + 2\alpha\int_0^t\int_0^\infty \big(u_{xx}^2+c_0v_{xx}^2\big)\,dx\,d\tau \\
&\leqslant (1+c_0)\delta_0 + \frac{\varepsilon\alpha}{2}\int_0^t\int_0^\infty u_{xx}^2\,dx\,d\tau + \frac{2}{\varepsilon\alpha}\int_0^t\int_0^\infty v_x^2\,dx\,d\tau + \frac{1}{2\varepsilon(1-\alpha)}\int_0^t\int_0^\infty v_{xx}^2\,dx\,d\tau \\
&\quad + 2\nu^2c_0^2\varepsilon(1-\alpha)\int_0^t\int_0^\infty u_x^2\,dx\,d\tau + \frac{\varepsilon\alpha}{2}\int_0^t\int_0^\infty u_{xx}^2\,dx\,d\tau + \frac{2}{\varepsilon\alpha}\int_0^t\int_0^\infty \bar\theta_x^2\,dx\,d\tau \\
&\quad + \frac{1}{2\varepsilon(1-\alpha)}\int_0^t\int_0^\infty v_{xx}^2\,dx\,d\tau + 8c_0^2\varepsilon(1-\alpha)\int_0^t\int_0^\infty u^2v_x^2\,dx\,d\tau \\
&\quad + (\delta+\delta_0)^{\frac12}\int_0^t\int_0^\infty v_{xx}^2\,dx\,d\tau + C(\delta+\delta_0)^{-\frac12}\int_0^t\int_0^\infty \bar\psi^2v_x^2\,dx\,d\tau
\end{aligned}$$

$$+ 2c_0\|\bar\theta_x\|_{L^\infty([0,T]\times\mathbb{R}_+)}\int_0^t\int_0^\infty\left(u^2+v_{xx}^2\right)dx\,d\tau + c_0\delta\int_0^t\int_0^\infty v_{xx}^2\,dx\,d\tau + \frac{c_0}{\delta}\int_0^t\int_0^\infty F^2(x,\tau)\,dx\,d\tau. \tag{3.27}$$

From (3.17), (3.23), (3.24) and Lemma 2.2, we have

$$\begin{aligned}&\int_0^\infty\left(u_x^2+c_0v_x^2\right)dx + 2(1-\alpha)\int_0^t\int_0^\infty\left(u_x^2+c_0v_x^2\right)dx\,d\tau + 2\alpha\int_0^t\int_0^\infty\left(u_{xx}^2+c_0v_{xx}^2\right)dx\,d\tau\\
&\quad\leqslant C(\delta+\delta_0)+\varepsilon\alpha\int_0^t\int_0^\infty u_{xx}^2\,dx\,d\tau + \left\{\frac{1}{\varepsilon(1-\alpha)}+C(\delta+\delta_0)^{\frac12}\right\}\int_0^t\int_0^\infty v_{xx}^2\,dx\,d\tau\\
&\qquad + C\int_0^t\int_0^\infty\left(u^2+u_x^2+v_x^2\right)dx\,d\tau + C(\delta+\delta_0)^{-\frac12}\int_0^t\int_0^\infty v_x^2\,dx\,d\tau.\end{aligned} \tag{3.28}$$

With the help of Lemma 3.4, (3.28) gives

$$\begin{aligned}&\int_0^\infty\left(u_x^2+c_0v_x^2\right)dx + (2-\varepsilon)\alpha\int_0^t\int_0^\infty u_{xx}^2\,dx\,d\tau\\
&\quad + \left\{2c_0\alpha-\frac{1}{\varepsilon(1-\alpha)}-C(\delta+\delta_0)^{\frac12}\right\}\int_0^t\int_0^\infty v_{xx}^2\,dx\,d\tau \leqslant C(\delta+\delta_0)^{\frac12}.\end{aligned} \tag{3.29}$$

Thus the proof of Lemma 3.5 is completed by (3.18) and (3.29) provided that δ and δ_1 are sufficiently small. □

Now we are in a position to complete the proof of the global existence of solutions to the problem (3.2)–(3.4). By the standard argument, the local existence and uniform a priori estimates yield the global existence. In fact, let δ and δ_1 sufficiently small such that we can find $C_1>0$ and $C_2>0$ independent of any $T>0$ to make Lemmas 3.4 and 3.5 hold. Furthermore let δ and δ_0 sufficiently small such that

$$\max\left\{2\delta_0^2, C_1(\delta+\delta_0)+C_2(\delta+\delta_0)^{\frac12}\right\}<\delta_1^2.$$

Define

$$T^* = \sup\left\{T\colon \sum_{k=0}^{1}\left(\left\|\partial_x^k u(t)\right\|^2+\left\|\partial_x^k v(t)\right\|^2\right)\leqslant\delta_1^2 \text{ for any } 0\leqslant t\leqslant T\right\}.$$

Thus it follows from Lemma 3.3 that $0<T^*\leqslant\infty$. If $T^*<\infty$, then by Lemmas 3.4 and 3.5, we have that for any $0\leqslant t\leqslant T^*$,

$$\sum_{k=0}^{1}\left(\left\|\partial_x^k u(t)\right\|^2+\left\|\partial_x^k v(t)\right\|^2\right)\leqslant C_1(\delta+\delta_0)+C_2(\delta+\delta_0)^{\frac12}<\delta_1^2,$$

which is a contradiction with the definition of T^*. So there exist global solutions (u, v) to the problem (3.2)–(3.4) such that

$$\sum_{k=0}^{1}\left(\left\|\partial_x^k u(t)\right\|^2+\left\|\partial_x^k v(t)\right\|^2\right)\leqslant \delta_1^2, \quad 0\leqslant t<\infty.$$

Again by Lemmas 3.4–3.5, we have (3.7).

Finally, we prove that (3.8) is true. For this purpose, we introduce the following lemma proved in [13].

Lemma 3.6. *If $g(t)\geqslant 0$, $g(t)\in L^1(0,\infty)$ and $g'(t)\in L^1(0,\infty)$, then $g(t)\to 0$ as $t\to\infty$.*

Taking $g(t)=\|u_x(t)\|^2$ in Lemma 3.6, we can conclude from (3.7) that $g(t)\in L^1(0,\infty)$. Denote the inner product by $\langle\cdot,\cdot\rangle$ in $L^2(\mathbb{R}_+)$. By using the definition of L^2-inner product and integrating by parts, we have that $g'(t)=2\langle u_x,u_{xt}\rangle=-2\langle u_t,u_{xx}\rangle$. It is easy to verify from (3.7) that

$$-\langle u_t,u_{xx}\rangle=\langle -u_t,u_{xx}\rangle=\big\langle(1-\alpha)u+v_x-\alpha u_{xx}+\bar{\theta}_x,u_{xx}\big\rangle\in L^1(0,\infty).$$

Hence, $g'(t)\in L^1(0,\infty)$, and

$$g(t)=\left\|u_x(t)\right\|^2\to 0 \quad \text{as } t\to\infty. \tag{3.30}$$

Applying the Sobolev inequality, we have from (3.30) and (3.7) that

$$\sup_{x\in\mathbb{R}_+}\left|u(x,t)\right|\leqslant \left\|u(t)\right\|^{\frac12}\left\|u_x(t)\right\|^{\frac12}\to 0 \quad \text{as } t\to\infty. \tag{3.31}$$

The same argument can be applied to $v(x,t)$, i.e.,

$$\sup_{x\in\mathbb{R}_+}\left|v(x,t)\right|\leqslant \left\|v(t)\right\|^{\frac12}\left\|v_x(t)\right\|^{\frac12}\to 0 \quad \text{as } t\to\infty. \tag{3.32}$$

This ends the proof of (3.8).

4. Decay rate

In this section, we will prove (3.9) in Theorem 3.1. First let us consider L^2 decay rates of solutions to the problem (3.2)–(3.4) under the following a priori assumption:

$$\sup_{0\leqslant t\leqslant T}\left\{e^{lt}\sum_{k=0}^{1}\left(\left\|\partial_x^k u(t)\right\|^2+\left\|\partial_x^k v(t)\right\|^2\right)\right\}\leqslant 1, \tag{4.1}$$

with

$$l=\min\left\{(2-\varepsilon)(1-\alpha),\,2(1-\alpha)-\frac{c_0\nu^2}{\varepsilon\alpha}\right\}, \tag{4.2}$$

where ε and c_0 are defined by (3.18). Our aim is to prove uniform estimates

$$e^{lt}\sum_{k=0}^{1}\left(\left\|\partial_x^k u(t)\right\|^2+\left\|\partial_x^k v(t)\right\|^2\right)\leqslant \frac12, \quad 0\leqslant t\leqslant T,$$

provided that δ and δ_0 are sufficiently small. Thus by the same contradiction argument as in the last section, we obtain the L^2 decay rates of solutions (u,v). Hence we omit its details for brevity and only prove the above uniform estimates.

By the Sobolev inequality, we have from (4.1) that

$$\|(u,v)\|_{L^\infty} \leqslant e^{-\frac{l}{2}t}. \tag{4.3}$$

The following Gronwall's inequality will be also used later on.

Lemma 4.1 *(Gronwall's inequality). Let $\eta(\cdot)$ be a nonnegative continuous function on $[0,\infty)$, which satisfies the differential inequality*

$$\eta'(t) + \lambda\eta(t) \leqslant \omega(t), \quad 0 \leqslant t < \infty,$$

where λ is a positive constant and $\omega(t)$ is a nonnegative continuous function on $[0,\infty)$. Then

$$\eta(t) \leqslant \left(\eta(0) + \int_0^t e^{\lambda\tau}\omega(\tau)\,d\tau\right)e^{-\lambda t}, \quad 0 \leqslant t < \infty. \tag{4.4}$$

Now we can state the main result about L^2 decay rates of solutions.

Theorem 4.2. *Suppose that $(u,v)(x,t)$ are solutions to the problem (3.2)–(3.4) under the assumptions imposed in Theorem 3.1. If δ and δ_0 are sufficiently small, then we have that for any $0 \leqslant t \leqslant T$,*

$$\|\partial_x^k u(t)\|^2 + \|\partial_x^k v(t)\|^2 \leqslant C_3(\delta+\delta_0)^{\frac{1}{8}}e^{-lt}, \quad k = 0, 1, \tag{4.5}$$

where C_3 is a positive constant depending only on α and ν and l is defined by (4.2).

Proof. The proof consists of two steps.

Step 1. Taking $(3.2)_1 \times 2u + (3.2)_2 \times 2c_0 v$ and integrating over $x \in \mathbb{R}_+$, we reach at

$$\begin{aligned}
&\frac{d}{dt}\int_0^\infty (u^2 + c_0v^2)\,dx + 2(1-\alpha)\int_0^\infty (u^2 + c_0v^2)\,dx + 2\alpha\int_0^\infty (u_x^2 + c_0v_x^2)\,dx \\
&\quad = -2\int_0^\infty uv_x\,dx + 2\nu c_0\int_0^\infty vu_x\,dx - 2\int_0^\infty u\bar{\theta}_x\,dx - 2c_0\int_0^\infty (u_x + \bar{\psi}_x)v^2\,dx \\
&\qquad + 4c_0\int_0^\infty \bar{\theta}_x uv\,dx + 2c_0\int_0^\infty vF(x,t)\,dx \\
&\quad \leqslant \varepsilon(1-\alpha)\int_0^\infty u^2\,dx + \frac{1}{\varepsilon(1-\alpha)}\int_0^\infty v_x^2\,dx + \varepsilon\alpha\int_0^\infty u_x^2\,dx + \frac{c_0^2\nu^2}{\varepsilon\alpha}\int_0^\infty v^2\,dx - 2\int_0^\infty u\bar{\theta}_x\,dx \\
&\qquad + 2c_0\|\bar{\theta}_x\|_{L^\infty}\int_0^\infty u^2\,dx + 2c_0\big(\|\bar{\psi}_x\|_{L^\infty} + \|\bar{\theta}_x\|_{L^\infty}\big)\int_0^\infty v^2\,dx \\
&\qquad - 2c_0\int_0^\infty u_xv^2\,dx + 2c_0\int_0^\infty vF(x,t)\,dx.
\end{aligned} \tag{4.6}$$

By Lemma 2.2, rearrangement of the terms in the inequality gives

$$\begin{aligned}
&\frac{d}{dt}\int_0^\infty \left(u^2+c_0v^2\right)dx+(2-\varepsilon)(1-\alpha)\int_0^\infty u^2\,dx+\left\{2(1-\alpha)-\frac{c_0v^2}{\varepsilon\alpha}\right\}\int_0^\infty c_0v^2\,dx\\
&\qquad+(2-\varepsilon)\alpha\int_0^\infty u_x^2\,dx+\left\{2c_0\alpha-\frac{1}{\varepsilon(1-\alpha)}\right\}\int_0^\infty v_x^2\,dx\\
&\quad\leqslant C\delta e^{-(1-\alpha)t}\int_0^\infty \left(u^2+c_0v^2\right)dx-2\int_0^\infty u\bar{\theta}_x\,dx-2c_0\int_0^\infty u_xv^2\,dx\\
&\qquad+2c_0\int_0^\infty vF(x,t)\,dx. \qquad (4.7)
\end{aligned}$$

Next, we estimate terms in the right-hand side of (4.7).

In fact, we have from the assumption (4.1) and Lemma 2.2

$$\begin{aligned}
-2\int_0^\infty u\bar{\theta}_x\,dx &\leqslant \delta e^{\{\frac{l}{2}-(1-\alpha)\}t}\int_0^\infty u^2\,dx+\frac{1}{\delta}e^{-\{\frac{l}{2}-(1-\alpha)\}t}\int_0^\infty \bar{\theta}_x^2\,dx\\
&\leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t}. \qquad (4.8)
\end{aligned}$$

Moreover, we get from (3.7) by Sobolev inequality

$$\left\|(u,v)\right\|_{L^\infty}\leqslant C(\delta+\delta_0)^{\frac{1}{4}}. \qquad (4.9)$$

We derive by the Cauchy–Schwarz inequality from (4.1), (4.3) and (4.9)

$$\begin{aligned}
-2c_0\int_0^\infty u_xv^2\,dx &\leqslant c_0\left(\int_0^\infty u_x^2|v|\,dx+\int_0^\infty |v|^3\,dx\right)\\
&\leqslant c_0\|v\|_{L^\infty}^{\frac{1}{2}}\|v\|_{L^\infty}^{\frac{1}{2}}\int_0^\infty \left(u_x^2+v^2\right)dx\\
&\leqslant C(\delta+\delta_0)^{\frac{1}{8}}e^{-\frac{5l}{4}t}. \qquad (4.10)
\end{aligned}$$

In addition, again from the assumption (4.1) and Lemma 2.2, we deduce

$$\begin{aligned}
2c_0\int_0^\infty vF(x,t)\,dx &\leqslant \delta e^{\{\frac{l}{2}-(1-\alpha)\}t}\int_0^\infty v^2\,dx+\frac{c_0^2}{\delta}e^{-\{\frac{l}{2}-(1-\alpha)\}t}\int_0^\infty F^2(x,t)\,dx\\
&\leqslant \delta e^{\{\frac{l}{2}-(1-\alpha)\}t}e^{-lt}+\frac{C}{\delta}e^{-\{\frac{l}{2}-(1-\alpha)\}t}\int_0^\infty \left(\bar{\psi}_x^2+\bar{\theta}_x^2\right)dx\\
&\leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t}. \qquad (4.11)
\end{aligned}$$

Thus, we get from (4.7), (4.8), (4.10) and (4.11) that

$$\frac{d}{dt}\int_0^\infty \big(u^2+c_0v^2\big)\,dx+(2-\varepsilon)(1-\alpha)\int_0^\infty u^2\,dx+\left\{2(1-\alpha)-\frac{c_0\nu^2}{\varepsilon\alpha}\right\}\int_0^\infty c_0v^2\,dx \leqslant C(\delta+\delta_0)^{\frac{1}{8}}e^{-\frac{5l}{4}t}+C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t}. \tag{4.12}$$

Recalling the definition of l in (4.2), we have

$$\frac{d}{dt}\int_0^\infty \big(u^2+c_0v^2\big)\,dx+l\int_0^\infty \big(u^2+c_0v^2\big)\,dx\leqslant C(\delta+\delta_0)^{\frac{1}{8}}e^{-\frac{5l}{4}t}+C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t}. \tag{4.13}$$

Noticing that $\frac{l}{2}<1-\alpha$, we obtain from Lemma 4.1

$$\int_0^\infty \big(u^2+c_0v^2\big)\,dx\leqslant C(\delta+\delta_0)^{\frac{1}{8}}e^{-lt}, \tag{4.14}$$

which implies (4.5) for $k=0$.

Step 2. Taking $(3.2)_1\times(-2u_{xx})+(3.2)_2\times(-2c_0v_{xx})$ and integrating with respect to x over $\mathbb{R}_+$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^\infty \big(u_x^2+c_0v_x^2\big)\,dx+2(1-\alpha)\int_0^\infty \big(u_x^2+c_0v_x^2\big)\,dx+2\alpha\int_0^\infty \big(u_{xx}^2+c_0v_{xx}^2\big)\,dx\\
&\quad\leqslant \varepsilon(1-\alpha)c_0\int_0^\infty v_x^2\,dx+\frac{1}{\varepsilon(1-\alpha)c_0}\int_0^\infty u_{xx}^2\,dx+\frac{c_0\nu^2}{\varepsilon\alpha}\int_0^\infty u_x^2\,dx+\varepsilon\alpha c_0\int_0^\infty v_{xx}^2\,dx\\
&\qquad+2\int_0^\infty u_{xx}\bar\theta_x\,dx-4c_0\int_0^\infty uv_xv_{xx}\,dx-4c_0\int_0^\infty \bar\psi v_xv_{xx}\,dx-4c_0\int_0^\infty \bar\theta_x uv_{xx}\,dx\\
&\qquad-2c_0\int_0^\infty v_{xx}F(x,t)\,dx.
\end{aligned} \tag{4.15}$$

From Lemma 2.2, it may be shown that

$$\begin{aligned}
2\int_0^\infty u_{xx}\bar\theta_x\,dx&\leqslant \delta e^{\{\frac{l}{2}-(1-\alpha)\}t}\int_0^\infty u_{xx}^2\,dx+\frac{1}{\delta}e^{-\{\frac{l}{2}-(1-\alpha)\}t}\int_0^\infty \bar\theta_x^2\,dx\\
&\leqslant \delta\int_0^\infty u_{xx}^2+C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t}.
\end{aligned} \tag{4.16}$$

From (4.3) and (4.9), we have

$$
\begin{aligned}
-4c_0\int_0^\infty uv_xv_{xx}\,dx &\leqslant (\delta+\delta_0)^{\frac14}\int_0^\infty v_{xx}^2\,dx+4c_0^2(\delta+\delta_0)^{-\frac14}\int_0^\infty u^2v_x^2\,dx\\
&\leqslant (\delta+\delta_0)^{\frac14}\int_0^\infty v_{xx}^2\,dx+C(\delta+\delta_0)^{-\frac14}\|u\|_{L^\infty(\mathbb{R}_+)}^{\frac32}\|u\|_{L^\infty(\mathbb{R}_+)}^{\frac12}\int_0^\infty v_x^2\,dx\\
&\leqslant (\delta+\delta_0)^{\frac14}\int_0^\infty v_{xx}^2\,dx+C(\delta+\delta_0)^{\frac18}e^{-\frac54 lt}. \qquad (4.17)
\end{aligned}
$$

Again using (4.1) and Lemma 2.2, we get

$$
\begin{aligned}
-4c_0\int_0^\infty \bar\psi v_xv_{xx}\,dx &\leqslant (\delta+\delta_0)^{\frac14}\int_0^\infty v_{xx}^2\,dx+4c_0^2(\delta+\delta_0)^{-\frac14}\int_0^\infty \bar\psi^2v_x^2\,dx\\
&\leqslant (\delta+\delta_0)^{\frac14}\int_0^\infty v_{xx}^2\,dx+C(\delta+\delta_0)^{\frac14}e^{-2(1-\alpha)t} \qquad (4.18)
\end{aligned}
$$

and

$$
\begin{aligned}
-4c_0\int_0^\infty \bar\theta_x uv_{xx}\,dx &\leqslant \delta\int_0^\infty v_{xx}^2\,dx+4c_0^2\delta^{-1}\int_0^\infty \bar\theta_x^2u^2\,dx\\
&\leqslant \delta\int_0^\infty v_{xx}^2\,dx+C\delta e^{-\{2(1-\alpha)+l\}t}. \qquad (4.19)
\end{aligned}
$$

Moreover, we have

$$
-2c_0\int_0^\infty v_{xx}F\,dx\leqslant \delta\int_0^\infty v_{xx}^2\,dx+4c_0^2\delta^{-1}\int_0^\infty F^2\,dx\leqslant \delta\int_0^\infty v_{xx}^2\,dx+C\delta e^{-2(1-\alpha)t}. \qquad (4.20)
$$

Substituting (4.16)–(4.20) into (4.15) and noticing that

$$
-\bigl\{2(1-\alpha)+l\bigr\}<-\left\{\frac{l}{2}+(1-\alpha)\right\},\qquad -2(1-\alpha)<-\left\{\frac{l}{2}+(1-\alpha)\right\},
$$

we have

$$
\begin{aligned}
&\frac{d}{dt}\int_0^\infty\bigl(u_x^2+c_0v_x^2\bigr)\,dx+\left\{2(1-\alpha)-\frac{c_0\nu^2}{\varepsilon\alpha}\right\}\int_0^\infty u_x^2\,dx+(2-\varepsilon)(1-\alpha)\int_0^\infty c_0v_x^2\,dx\\
&\qquad+\left\{2\alpha-\frac{1}{\varepsilon(1-\alpha)c_0}-\delta\right\}\int_0^\infty u_{xx}^2\,dx+\bigl\{(2-\varepsilon)\alpha c_0-2(\delta+\delta_0)^{\frac14}-2\delta\bigr\}\int_0^\infty v_{xx}^2\,dx\\
&\quad\leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t}+C(\delta+\delta_0)^{\frac18}e^{-\frac54 lt}+C(\delta+\delta_0)^{\frac14}e^{-2(1-\alpha)t}. \qquad (4.21)
\end{aligned}
$$

Together with (4.2), (4.21) gives

$$\frac{d}{dt}\int_0^\infty \left(u_x^2 + c_0 v_x^2\right) dx + l\int_0^\infty \left(u_x^2 + c_0 v_x^2\right) dx$$
$$\leqslant C\delta e^{-\{\frac{l}{2}+(1-\alpha)\}t} + C(\delta+\delta_0)^{\frac{1}{8}} e^{-\frac{5}{4}lt} + C(\delta+\delta_0)^{\frac{1}{4}} e^{-2(1-\alpha)t}. \tag{4.22}$$

Thus we have from Lemma 4.1 that

$$\int_0^\infty \left(u_x^2 + c_0 v_x^2\right) dx \leqslant C(\delta+\delta_0)^{\frac{1}{8}} e^{-lt}, \tag{4.23}$$

which implies (4.5) for $k=1$. The proof of Theorem 4.2 is completed. □

Remark. Immediately it follows from Theorem 4.2 that

$$\sum_{k=0}^{1}\left(\left\|\partial_x^k u(t)\right\|^2 + \left\|\partial_x^k v(t)\right\|^2\right) \leqslant e^{-lt}, \quad 0 \leqslant t < \infty.$$

In fact the same contradiction argument as in Section 3 can be applied. For brevity, we omit its proof.

Finally, we devote ourselves to the proof of L^p $(p \geqslant 2)$ decay rates (3.9) by pointwise estimates. For the case when $k=0$, it follows from the Gagliardo–Nirenberg inequality that for any $t \geqslant 0$,

$$\left\|u(t)\right\|_{L^p} \leqslant C\left\|u(t)\right\|_{L^2}^{\frac{p+2}{2p}} \left\|\partial_x u(t)\right\|_{L^2}^{\frac{p-2}{2p}} \leqslant Ce^{-\frac{l}{2}t}. \tag{4.24}$$

Similarly we have that for any $t \geqslant 0$,

$$\left\|v(t)\right\|_{L^p} \leqslant Ce^{-\frac{l}{2}t}. \tag{4.25}$$

For the case when $k=1$, we need the following inequality.

Lemma 4.3 *(Young's inequality). If $f \in L^p$, $g \in L^r$, $p \geqslant 1$, $r \geqslant 1$ and $\frac{1}{p}+\frac{1}{r} \geqslant 1$, then $h = f * g \in L^q$, where $\frac{1}{q} = \frac{1}{p}+\frac{1}{r}-1$. Furthermore, it holds that*

$$\|h\|_{L^q} \leqslant \|f\|_{L^p} \cdot \|g\|_{L^r}.$$

From (3.14), we have that

$$\begin{cases} \partial_x u(x,t) = (\partial_x K(t) * u_0)(x) + \int_0^t \partial_x K(t-s) * (-v_x - \bar\theta_x)(\cdot,s)\,ds, \\ \partial_x v(x,t) = (\partial_x K(t) * v_0)(x) \\ \qquad\qquad + \int_0^t \partial_x K(t-s) * (\nu u_x + 2uv_x + 2\bar\psi v_x + 2\bar\theta_x u + F)(\cdot,s)\,ds. \end{cases} \tag{4.26}$$

Then it holds that

$$\left\|\partial_x u(t)\right\|_{L^p} \leqslant \left\|\partial_x K(t) * u_0\right\|_{L^p} + \int_0^t \left\|\partial_x K(t-s) * v_x\right\|_{L^p} ds$$
$$+ \int_0^t \left\|\partial_x K(t-s) * \bar\theta_x\right\|_{L^p} ds \tag{4.27}$$

and

$$
\begin{aligned}
\|\partial_x v(t)\|_{L^p} \leqslant{} & \|\partial_x K(t) * v_0\|_{L^p} + \int_0^t \|\partial_x K(t-s) * \nu u_x\|_{L^p}\,ds \\
& + \int_0^t \|\partial_x K(t-s) * (2uv_x + 2\bar{\psi} v_x + 2\bar{\theta}_x u)\|_{L^p}\,ds \\
& + \int_0^t \|\partial_x K(t-s) * (\nu \bar{\psi}_x + 2\bar{\psi}\bar{\theta}_x)\|_{L^p}\,ds. \qquad (4.28)
\end{aligned}
$$

It follows from the Young's inequality that

$$
\|\partial_x K(t) * u_0\|_{L^p} \leqslant \|\partial_x K(t)\|_{L^{\frac{2p}{p+2}}} \|u_0\|_{L^2}.
$$

Other terms can be similarly dealt with. Thus by using Lemma 2.2 and Theorem 4.2, we have from (4.27) and (4.28) that

$$
\begin{aligned}
\|\partial_x u(t)\|_{L^p} \leqslant{} & Ce^{-(1-\alpha)t} t^{-\frac{3}{4}+\frac{1}{2p}} + C\int_0^t e^{-(1-\alpha)(t-s)}(t-s)^{-\frac{3}{4}+\frac{1}{2p}} e^{-\frac{l}{2}s}\,ds \\
& + C\int_0^t e^{-(1-\alpha)(t-s)}(t-s)^{-\frac{3}{4}+\frac{1}{2p}} e^{-(1-\alpha)s}(1+s)^{-\frac{1}{4}}\,ds \qquad (4.29)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\partial_x v(t)\|_{L^p} \leqslant{} & Ce^{-(1-\alpha)t} t^{-\frac{3}{4}+\frac{1}{2p}} + C\int_0^t e^{-(1-\alpha)(t-s)}(t-s)^{-\frac{3}{4}+\frac{1}{2p}} e^{-\frac{l}{2}s}\,ds \\
& + C\int_0^t e^{-(1-\alpha)(t-s)}(t-s)^{-\frac{3}{4}+\frac{1}{2p}} e^{-ls}\,ds \\
& + C\int_0^t e^{-(1-\alpha)(t-s)}(t-s)^{-\frac{3}{4}+\frac{1}{2p}} e^{-(1-\alpha)s}(1+s)^{-\frac{1}{4}}\,ds. \qquad (4.30)
\end{aligned}
$$

Next we only prove that

$$
\int_0^t e^{-(1-\alpha)(t-s)}(t-s)^{-\frac{3}{4}+\frac{1}{2p}} e^{-\frac{l}{2}s}\,ds \leqslant Ce^{-\frac{l}{2}t}.
$$

Other integrals can be controlled in the same way. In fact, since $1-\alpha > l/2$, it holds that

$$\begin{aligned}\int_0^t e^{-(1-\alpha)(t-s)}(t-s)^{-\frac{3}{4}+\frac{1}{2p}}e^{-\frac{l}{2}s}\,ds &= e^{-\frac{l}{2}t}\int_0^t e^{-(1-\alpha-\frac{l}{2})x}x^{-\frac{3}{4}+\frac{1}{2p}}\,dx\\ &\leqslant Ce^{-\frac{l}{2}t}\int_0^\infty x^{-\frac{3}{4}+\frac{1}{2p}}e^{-x}\,dx\\ &\leqslant Ce^{-\frac{l}{2}t}.\end{aligned}$$

Thus it follows from (4.29) and (4.30) that for any $t \geqslant \tau > 0$,

$$\left\|\partial_x u(t)\right\|_{L^p} \leqslant Ce^{-(1-\alpha)t}t^{-\frac{3}{4}+\frac{1}{2p}} + Ce^{-\frac{l}{2}t} + Ce^{-(1-\alpha)t}t^{\frac{1}{4}+\frac{1}{2p}} \leqslant C(\tau)e^{-\frac{l}{2}t} \tag{4.31}$$

and

$$\begin{aligned}\left\|\partial_x v(t)\right\|_{L^p} &\leqslant Ce^{-(1-\alpha)t}t^{-\frac{3}{4}+\frac{1}{2p}} + Ce^{-\frac{l}{2}t} + Ce^{-\min\{1-\alpha,l\}t}t^{\frac{1}{4}+\frac{1}{2p}} + Ce^{-(1-\alpha)t}t^{\frac{1}{4}+\frac{1}{2p}}\\ &\leqslant C(\tau)e^{-\frac{l}{2}t}.\end{aligned} \tag{4.32}$$

Hence combining (4.24), (4.25), (4.31) and (4.32) yields (3.9). The proof of Theorem 3.1 is completed.

5. Discussions

In this paper, we have established the global existence for the initial boundary value problem of system (1.1) on a quadrant, and the exponential decay rates of deviation from diffusion waves. This approach applies equally well to the general system (1.4) [15,16]. As a matter of fact, as long as dissipation terms dominate, diffusion waves are stable in general. Furthermore, our approach also applies to a companion system of (1.4), called *conservative form* in [7]:

$$\begin{cases}\psi_t = -(\sigma-\alpha)\psi - \sigma\theta_x + \alpha\psi_{xx},\\ \theta_t = -(1-\beta)\theta + \nu\psi_x + (\psi\theta)_x + \beta\theta_{xx}.\end{cases} \tag{5.1}$$

Though numerical results have demonstrated the drastic difference between system (5.1) and (1.4), in dissipation dominating regime of parameters, both systems converge to diffusion waves in the same manner.

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